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ADDITIVE COMPLETION OF LACUNARY SEQUENCES

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To the memory of Pál Erdős

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Foreword

Thirty years ago I read the following question of Erdős [4]:

"Does there exist a sequence $a_1 < a_2 < \dots$ with $A(x) < c\frac{x}{\log x}$ $\left(A(x) = \sum_{a_i<x} 1\right)$ so that every sufficiently large number is of the form $2^k + a_i$? \$10"

I sent my solution to Erdős in a letter (in Hungarian). He translated my letter into English and sent it to the Canadian Math. Bulletin; this became my first paper to appear.

In this paper we will find, among others, the best value of the constant c in the above question, which was also asked by Erdős.

1. Introduction

We call two sets A, B of positive integers *additive complements*, if their sum

$$A + B = \{a + b : a \in A, b \in B\}$$

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contains all sufficiently large integers. A pair of additive complements obviously satisfies

$$A(x)B(x) \geq x - K \tag{1.1}$$

with some constant K, where $A(x)$ denotes the number of elements of A up to x. Disproving a conjecture of Hanani, Danzer [2] constructed sets satisfying

$$A(x)B(x) \sim x. \tag{1.2}$$

For sets satisfying (1.2), Sárközy and Szemerédi [11] strengthened (1.1) to

$$A(x)B(x) - x \to \infty.$$

Narkiewicz [7] proved that whenever (1.2) holds, one of the sets is very thin and the other is dense, namely, we have

$$A(2x)/A(x) \to 1, \tag{1.3}$$

whence $A(x) = x^{o(1)}$ and $B(x) = x^{1+o(1)}$, or these hold with the roles of A and B interchanged.

A weaker property that a pair of additive complements may have is

$$A(x)B(x) = O(x). \tag{1.4}$$

We call additive complements satisfying (1.4) *economic*, and those satisfying the stronger requirement (1.2) *exact*.

Sets given by polynomials are easily seen to have an economic completion, and by the above quoted result of Narkiewicz they never have an exact one. Lower bounds for the counting functions of a complement were studied by several authors. See Moser [6] for an early account, Cilleruelo [1] for the best known estimates, Habsieger and Ruzsa [5] for some related results.

An economic additive complement was constructed for the powers of 2 by Ruzsa [8], and for any linear recurrence set in Ruzsa [9]. In Ruzsa [10] it is proved that for any integer $a > 2$ the set of powers of a has even an exact complement. These constructions were based on special congruence properties of the set investigated. In particular, the construction of an exact complement for the powers of a required the existence of many integers m such that $m | a^m - 1$; no such integer $m > 1$ exists for $a = 2$.

Now we solve, by a different method, the last outstanding case.

Theorem 1. *The set of powers of 2 has an exact complement.*

All these results seem to indicate that the existence of an exact or economic complement depends on peculiar properties of a set. This is, however, not the case. We show that a wide class of sets, characterized by a growth condition, has exact additive complement.

Let $A=\{a_1,a_2,\dots\}$ be a set of positive integers, $1\le a_1<a_2<\dots$.

Theorem 2. *If*

$$\frac{a_{n+1}}{na_n}\to\infty,$$

then A has an exact complement.

Corollary 1.1. *If there is a constant k such that*

$$\frac{a_{n+k}}{na_n}\to\infty,$$

then A has an economic complement.

Corollary 1.2. *If there is a constant $c>0$ such that*

$$a_{n+1}>n^c a_n$$

for $n>n_0$, then A has an economic complement.

Corollary 1.3. *The set $A=\{(2n)!\}$ has an exact complement.*

Corollary 1.4. *The set $A=\{n!\}$ has an economic complement.*

I cannot decide whether the set of factorials has an exact complement.

2. Modular and global completion

We find a general criterion for the existence of an exact complement in terms of a similar property modulo m.

Let $m>a_1$ be an integer and write $k=A(m)$. Let $L(m)$ denote the smallest number l for which there are integers $b_1,\dots,b_l$ such that the numbers

$$a_i+b_j,\quad 1\le i\le k,\ 1\le j\le l$$

contain every residue modulo m.

Theorem 3. *Let A be a set satisfying Narkiewicz's condition (1.3). The following are equivalent:*

(a) A has an exact complement;
(b) $A(m)L(m)/m\to 1$;
(c) there is a sequence $m_1<m_2<\dots$ of positive integers such that $A(m_{i+1})/A(m_i)\to 1$ and $A(m_i)L(m_i)/m_i\to 1$.

A similar but more specialized connection was proved and applied in [10].

Proof. Clearly (a) $\Rightarrow$(b)$\Rightarrow$(c). We prove (c) $\Rightarrow$(a).

Condition (1.3) implies that for any sequences $x_i, y_i \to \infty$ such that $c < x_i/y_i < C$ for all i with some constants $0 < c < C$ we have $A(x_i) \sim A(y_i)$. This property will be used in the course of the proof for several choices of the sequences without further explanation.

Next we find an increasing sequence q_j of positive integers tending to infinity such that $A(q_j m_j) \sim A(m_j)$. A possible definition of q_j is the following: let q_j be the largest positive integer with the property that the inequality $A(qx) \le (1+1/q)A(x)$ holds for all $x \ge m_j$. This sequence is clearly increasing; if it did not tend to infinity, it would become constant, say q, after a point. This would mean the existence of arbitrarily large values of x such that $A((q+1)x) > (1+1/(q+1))A(x)$, in contradiction with the previous property.

We may also assume, by selecting a subsequence if necessary, that $m_{j+1}/m_j > 2$ for all j. Indeed, let m_j be in the selected subsequence and let k be the first subscript for which $m_k > 2m_j$. We select m_k next. We have $m_{k-1} \le 2m_j$, so $A(m_{k-1}) \sim A(m_j)$ by (1.3), $A(m_k) \sim A(m_{k-1})$ by assumption, so indeed $A(m_j) \sim A(m_k)$.

Now we define the set B in terms of these sequences m_j and q_j.

Write $A_j = A \cap [1, m_j]$, and take $U_j \subset [1, m_j]$ such that $|U_j| = L(m_j)$ and $A_j + U_j$ contains every residue modulo m_j. Put

$$V_j = U_j + \left\{ (q_j - 1)m_j, q_j m_j, (q_j + 1)m_j, \ldots, \left[\frac{q_{j+1} m_{j+1}}{m_j} \right] m_j \right\}.$$

Our exact complement will be

$$B = \bigcup V_j.$$

First we show that $A + B$ contains every sufficiently large integer.

Consider an integer n. If $n > n_0 = q_1 m_1$, then we can find a j such that

$$q_j m_j < n \le q_{j+1} m_{j+1}.$$

By definition there is an $a \in A_j$ and an $u \in U_j$ such that $a + u \equiv n \pmod{m_j}$. We claim that

$$n - a = u + \frac{n - a - u}{m_j} m_j \in V_j.$$

To see this we observe that

$$\frac{n - a - u}{m_j} \le \frac{q_{j+1} m_{j+1}}{m_j}$$

and

$$\frac{n-a-u}{m_j} > \frac{q_j m_j - 2m_j}{m_j} = q_j - 2,$$

so, being an integer, this quotient is $\geq q_j - 1$.

Now we estimate $B(x)$.

Let $q_j m_j < x \leq q_{j+1} m_{j+1}$. The set $B \cap [1,x]$ contains completely the sets $V_1, \dots, V_{j-1}$, an intitial segment of V_j and possibly an initial segment of V_{j+1}.

The cardinality of V_i is

$$|V_i| = |U_i| \left(\left[\frac{q_{i+1} m_{i+1}}{m_i} \right] - q_i + 2 \right) \leq |U_i| \left(\frac{q_{i+1} m_{i+1} - q_i m_i}{m_i} + 2 \right). \tag{2.1}$$

From V_j we have the elements up to x. They belong to $U_j + t m_j$ with $t \leq x/m_j$, thus their number can be estimated as

$$|V_j \cap [1,x]| \leq |U_j| \left(\left[\frac{x}{m_j} \right] - q_j + 2 \right) \leq |U_j| \left(\frac{x - q_j m_j}{m_j} + 2 \right). \tag{2.2}$$

Finally, V_{j+1} is the union of translates of U_{j+1} by certain numbers, the second of which is already $q_{j+1} m_{j+1} \geq x$, thus only the first can contain integers up to x. The first translate starts at $(q_{j+1} - 1) m_{j+1}$, consequently

$$|V_{j+1} \cap [1,x]| \leq \delta |U_{j+1}|, \tag{2.3}$$

where $\delta = 0$ if $x < (q_{j+1} - 1) m_{j+1}$, and $\delta = 1$ if $(q_{j+1} - 1) m_{j+1} \leq x \leq q_{j+1} m_{j+1}$.

On summing (2.1), (2.2) and (2.3) we obtain that

$$B(x) \leq \sum_{i=1}^{j-1} |U_i| \frac{q_{i+1} m_{i+1} - q_i m_i}{m_i} + |U_j| \frac{x - q_j m_j}{m_j} + 2 \sum_{i=1}^{j} |U_i| + \delta |U_{j+1}|, \tag{2.4}$$

with the δ described above.

Our aim is to deduce that $B(x) \leq (1 + o(1)) \dfrac{x}{A(x)}$.

To estimate the first sum in (2.4), first we show that for any $\varepsilon > 0$ there is a k such that

$$|U_i| \frac{q_{i+1} m_{i+1} - q_i m_i}{m_i} < (1 + \varepsilon) \left(\frac{q_{i+1} m_{i+1}}{A(m_{i+1})} - \frac{q_i m_i}{A(m_i)} \right) \tag{2.5}$$

for $i \geq k$; this will yield a nice telescopic sum. To achieve this it is sufficient to have $|U_i| < (1 + \eta) m_i / A(m_i)$ and

$$\frac{q_{i+1} m_{i+1} - q_i m_i}{A(m_i)} < (1 + \eta) \left(\frac{q_{i+1} m_{i+1}}{A(m_{i+1})} - \frac{q_i m_i}{A(m_i)} \right) \tag{2.6}$$

with $\eta = \varepsilon/3$, say. The first is among the assumptions. To show (2.6) we rearrange it as

$$\frac{A(m_i)}{A(m_{i+1})} > \frac{q_{i+1}m_{i+1} + \eta q_i m_i}{(1+\eta)q_{i+1}m_{i+1}}.$$

Since $q_i m_i \le q_{i+1} m_i < q_{i+1}m_{i+1}/2$, the right side here is at most $\dfrac{1+\eta/2}{1+\eta}$, while the left side tends to 1 by our assumptions, thus this inequality holds for large i.

By applying (2.5) we find that the first sum in (2.4) is at most

$$(1+\varepsilon)\sum_{i=k}^{j-1}\left(\frac{q_{i+1}m_{i+1}}{A(m_{i+1})} - \frac{q_i m_i}{A(m_i)}\right) + K_\varepsilon \le (1+\varepsilon)\frac{q_j m_j}{A(m_j)} + K_\varepsilon$$

with a certain constant K_ε depending on ε which accounts for the contribution of those initial terms for which (2.5) does not hold.

We estimate the second term of (2.4) by simply using $|U_j| < (1+\varepsilon)m_j/A(m_j)$. On adding this to the previous estimate we find that the first two terms together are $<(1+\varepsilon)x/A(m_j)+K_\varepsilon$. Since $A(x) \le A(q_{j+1}m_{j+1}) \sim A(m_{j+1}) \sim A(m_j)$, for $x > x_0(\varepsilon)$ the first two terms of (2.4) together are

$$\le (1+2\varepsilon))\frac{x}{A(x)} + K_\varepsilon \le (1+3\varepsilon))\frac{x}{A(x)}.$$

In the last step we used the fact that $A(x)/x \to 0$, which is a consequence of assumption (1.3). Since this is true for every positive ε, we can conclude that the first two terms of (2.4) together are $\le (1+o(1))\dfrac{x}{A(x)}$.

Next we demonstrate that the last sum in (2.4) is $o(x/A(x))$. Recall that

$$|U_i| = L(m_i) \sim \frac{m_i}{A(m_i)}.$$

Since $A(m_{i+1}) \sim A(m_i)$ amd $m_{i+1}/m_i \ge 2$, we have, with at most finitely many exceptions,

$$|U_{i+1}|/|U_i| > 3/2. \tag{2.7}$$

Hence for $i_0 < i \le j$ we have $|U_i| \le (2/3)^{j-i}|U_j|$ and so

$$2\sum_{i=1}^{j}|U_i| \le 2\sum_{i=1}^{j}(2/3)^{j-i}|U_j| + O(1) < 6|U_j| + O(1). \tag{2.8}$$

Now observe that

$$A(m_j) \sim A(m_{j+1}) \sim A(q_{j+1}m_{j+1}) \ge A(x).$$

As $|U_j| \sim m_j/A(m_j)$, this implies that the quantity in (2.8) is

$$\ll \frac{m_j}{A(x)} < \frac{1}{q_j}\frac{x}{A(x)} = o(x/A(x))$$

as wanted.

Finally we show that the last term is $o(x/A(x))$. This is clear when $\delta = 0$. If $\delta = 1$, then we use $|U_{j+1}| = L(m_{j+1}) \sim m_{j+1}/A(m_{j+1})$, $A(x) \le A(q_{j+1}m_{j+1}) \sim A(m_{j+1})$ and $x \ge (q_{j+1}-1)m_{j+1}$. These imply that $|U_{j+1}|A(x)/x \le (1+o(1))/(q_{j+1}-1) \to 0$.

This concludes the proof of Theorem 3.

3. General lacunary sequences

In this section we prove Theorem 2.

We start with some lemmas.

Lemma 3.1. *Let $b_0, b_1, \dots, b_{n-1}$ be positive integers, $\beta_0, \dots, \beta_{n-1}$ real numbers and $\delta_0, \dots, \delta_{n-1}$ positive numbers satisfying $b_j/b_{j-1} \ge (1+\delta_j)/\delta_{j-1}$ for all $j = 1, \dots, n-1$. Then there is an $\alpha \in (0,1)$ such that*

$$\{\alpha b_i - \beta_i\} < \delta_i \tag{3.1}$$

for all $0 \le i \le n-1$.

Proof. We shall select recursively intervals

$$I_0 \supset I_1 \supset \dots \supset I_{n-1}$$

such that $\lambda(I_j) = \delta_j/b_j$ and $\alpha \in I_j$ impies (3.1) for $0 = 1, \dots, j$. To this end we start with

$$I_0 = (\beta_0, \beta_0 + \delta_0/b_0)\,.$$

Assume that I_{j-1} is already defined. The numbers αb_j, $\alpha \in I_{j-1}$ cover an interval whose length is $\delta_{j-1}b_j/b_{j-1} \ge 1 + \delta_j$. Such an interval contains a subinterval of type $(r+\beta_j, r+\beta_j+\delta_j)$ with some integer r. Now we define the next interval by the inequalities

$$r + \beta_j \le \alpha b_j < r + \beta_j + \delta_j,$$

which guarantee (3.1) for $i = j$. ∎

In this section we will use the following special case.

Lemma 3.2. *Let $b_0, b_1, \dots, b_{n-1}$ be positive integers such that always $b_{i+1}/b_i \ge Kn$ with a number $K > 2$. Then there is an $\alpha \in (0,1)$ such that*

$$\frac{i}{n} \le \{\alpha b_i\} < \frac{i}{n} + \frac{2}{Kn} \tag{3.2}$$

for all $0 \le i \le n-1$. ∎

Lemma 3.3. *Let $b_0, b_1, \dots, b_{n-1}$ be positive integers, $K > 3$ and $\alpha \in (0,1)$ real numbers and $m > Kn \max b_i$ an integer. Assume that (3.2) holds for all $0 \le i \le n-1$. Then there is a set H of integers such that $\{b_0, \dots, b_{n-1}\} + H$ contains a complete residue system modulo m and*

$$|H| \le \left(1 + \frac{3}{K}\right)\frac{m}{n}. \tag{3.3}$$

Proof. Define r by the inequalities

$$\frac{r-1}{m} < \alpha \le \frac{r}{m}.$$

Then we have

$$0 \le \frac{r}{m} b_i - \alpha b_i < \frac{1}{m} b_i < \frac{1}{Kn}$$

and hence (3.2) yields

$$\frac{i}{n} \le \left\{\frac{r}{m} b_i\right\} < \frac{i}{n} + \frac{3}{Kn} \tag{3.4}$$

for all $0 \le i \le n-1$.

Clearly $1 \le r \le m$. We can exclude the possibility $r = m$, since in this case the fractional parts in (3.4) would vanish and (3.4) would not hold. So $1 \le r \le m-1$. Let r'/m' be the reduced form of r/m, and let d_i be the residue of $r' b_i$ modulo m'. (3.4) can be rewritten as

$$\frac{i}{n} \le \frac{d_i}{m'} < \frac{i}{n} + \frac{3}{Kn} \tag{3.5}$$

for $i = 0, \dots, n-1$. Define d_n by $d_n = d_1 + m'$; then (3.5) will hold for $i = n$ also and we have

$$d_0 < d_1 < \ldots < d_{n-1} < d_n = d_0 + m'.$$

Define $H^* = \{0, 1, \dots, t\}$, where $t = \max(d_{i+1} - d_i) - 1$. Clearly every integer is congruent to some $d_i + h$, $h \in H^*$ modulo m' and

$$|H^*| = \max(d_{i+1} - d_i) \le \left(\frac{1}{n} + \frac{3}{Kn}\right) m' = \left(1 + \frac{3}{K}\right)\frac{m'}{n}.$$

Define r^* by $r'r^* \equiv 1 \pmod{m'}$ and put $H' = \{r^*h : h \in H^*\}$. Since the numbers $d_i + h$, $h \in H^*$ cover every residue modulo m', so do the numbers $r^*(d_i + h) \equiv b_i + r^*h$. Thus $\{b_0, \dots, b_{n-1}\} + H'$ contains every residue modulo m' and

$$|H'| = |H^*| \le \left(1 + \frac{3}{K}\right) \frac{m'}{n}.$$

Now we put

$$H = H' + \left\{0, m', 2m', \dots, \left(\frac{m}{m'} - 1\right) m'.\right\}$$

If we add to a complete residue system modulo m' the numbers 0, m', $2m'$, …, $((m/m') - 1)m'$, we get a collection of m integers, pairwise incongruent modulo m, hence a complete residue system modulo m. This observation shows that $\{b_0, \dots, b_{n-1}\} + H$ contains every residue modulo m, and we have

$$|H| = \frac{m}{m'}|H'| \le \left(1 + \frac{3}{K}\right) \frac{m}{n}.$$

Proof of Theorem 2. We will apply Theorem 3. To this end we need to estimate $L(m)$.

Choose a (large) K. There is an i_0 such that the inequality $a_{i+1}/a_i > K^2$ holds for $i > i_0$. We are going to estimate $L(m)$ for values of m which are so large that $A(m) > Ki_0$.

Write $k = A(m)$. We will apply Lemmas 3.2 and 3.3 to the integers

$$a_{[k/K]+1}, a_{[k/K]+2}, \dots, a_k.$$

Their number is $n = k - [k/K] \ge k(1 - 1/K)$, and they satisfy $a_{i+1}/a_i \ge K^2 i \ge Kk > Kn$. The abovementioned lemmas yield that this set of integers has an additive completion H modulo m which satisfies

$$|H| \le \left(1 + \frac{3}{K}\right) \frac{m}{n} \le \left(1 + \frac{3}{K}\right) \frac{K}{K-1} \frac{m}{k}.$$

This is an upper estimate for $L(m)$. Since the coefficient of $m/k = m/A(m)$ can be made arbitrarily near to 1 by choosing K appropriately, we have established property (b) of Theorem 3, and we conclude that A has an exact complement.

4. Powers of 2

In this section we prove Theorem 1. This proof will be based on Lemmas 3.1 and 3.3, but they will be applied in a less straightforward way.

For any odd integer d let $r(d)$ denote the order of 2 modulo d, that is, the smallest positive integer k such that $d|2^k-1$.

Lemma 4.1. *Let n be an odd natural number. Define s by the formula*

$$s=\sum_{d|n}\frac{\phi(d)}{r(d)},$$

let $K>3$ be a real number and L an integer satisfying $2^L\geq 2Kn$. Write $N=n+Ls$. With these notations there are integers $b_0,\dots,b_{n-1}$ and a real number α such that
(a)these α and b_i satisfy (3.2),
(b)each b_i is of the form $b_i=2^{t_i}$ with some integer t_i satisfying $0\leq t_i<N$.

Proof. We split the residues $1,\dots,n$ modulo n into groups so that i and j are in the same group if $j\equiv 2^t i$ for some t. Clearly i and j can be in the same group only if $(i,n)=(j,n)$. For a given i with $(i,n)=d$, its group consists of $i,2i,\dots,2^{z-1}i$, where z is the smallest integer such that $2^z i\equiv i \pmod n$. This integer is clearly $z=r(n/d)$. Since there are altogether $\phi(n/d)$ residues such that $(i,n)=d$, the number of groups belonging to a fixed d is $\phi(n/d)/r(n/d)$. By summing this for all possible values of d we find that the total number of groups is just s.

Let $r_1,\dots,r_s$ be the cardinalities of these groups (in any order), and $l_1,\dots,l_s$ representatives of the corresponding groups. Write further $R_i=r_1+\dots+r_i$.

Now we define the numbers $b_i=2^{t_i}$. In view of the above grouping, there is a unique j, $1\leq j\leq s$, and a unique u, $0\leq u\leq r_j-1$, such that

$$i\equiv 2^u l_j \pmod n. \tag{4.1}$$

Given this j and u, we put

$$t_i=R_{j-1}+L(j-1)+u. \tag{4.2}$$

Our aim is to find an α that satisfies (3.2), which with our choice of b_i means

$$\left\{\alpha 2^{t_i}-\frac{i}{n}\right\}<\frac{2}{Kn}.$$

Substituting the values i and t_i from (4.1) and (4.2), this can be rewritten as

$$\left\{\alpha 2^{R_{j-1}+L(j-1)+u} - 2^u \frac{l_j}{n}\right\} < \frac{2}{Kn}. \tag{4.3}$$

For a given j, these inequalities have to hold for $0 \le u \le r_j - 1$. They are far from independent, and they all follow from the following stronger inequality for the case $u=0$:

$$\left\{\alpha 2^{R_{j-1}+L(j-1)} - \frac{l_j}{n}\right\} < \frac{1}{2^{r_j} Kn}. \tag{4.4}$$

(In fact, (4.4) is equivalent to these; but we need only the obvious fact that (4.4) implies (4.3) for all $0 \le u \le r_j - 1$.)

The existence of an α satisfying (4.4) follows from Lemma 3.1. We replace the n of the lemma by s, the numbers b_j by the numbers 2^{R_j+Lj} and δ_j by $1/(2^{r_{j+1}}Kn)$. The condition $b_j/b_{j-1} \ge (1+\delta_j)/\delta_{j-1}$ of the lemma becomes

$$2^{r_j+L} \ge 2^{r_j} Kn + 2^{r_j - r_{j+1}},$$

and this follows from our assumption $2^L \ge 2Kn$. By this the lemma is proved.

Proof of Theorem 1. To apply Lemma 4.1 we need to compute the s described there. This is difficult for general n, and the result may be too large for our purposes. We will consider some special values of n only.

Since 2 is a primitive root of 9 and 25, it is a primitive root of 3^a and 5^a for every a, that is, $r(3^a) = \phi(3^a) = 2 \cdot 3^{a-1}$ and $r(5^a) = \phi(5^a) = 4 \cdot 5^{a-1}$. Consequently $r(3^a 5^b) = \operatorname{lcm}[r(3^a), r(5^b)] = 4 \cdot 3^{a-1} 5^{b-1} = \phi(3^a 5^b)/2$ whenever $a, b \ge 1$. Hence if n is of the form $n = 3^a 5^b$, then $\phi(d)/r(d) \le 2$ for every $d|n$, thus

$$s \le \sum_{d|n} 2 = 2(a+1)(b+1) \le c(\log n)^2$$

with a suitable absolute constant c.

We are now going to estimate $L(m)$. Write $k = A(m) = 1 + [(\log m)/\log 2]$. Let n be the largest number of the form $n = 3^a 5^b < k - \sqrt{k}$. Since $(\log 3)/(\log 5)$ is irrational, the quotient of consecutive numbers of the form $3^a 5^b$ tends to 1, so $n/k \to 1$ as $k \to \infty$.

We put $K = n$ and take L to be the smallest integer satisfying $2^L > 2KN = 2n^2$. This integer satisfies $L \ll \log n$, thus $Ls \ll (\log n)^3$ and so $N = n + Ls < n + O((\log n)^3) < n + \sqrt{n} < k$ for large n.

Lemma 4.1 gives us certain integers $b_1,\dots,b_n$ which are all of the form 2^j, $j\le k$ and satisfy (3.2). Lemma 3.3 yields an additive complement H of these b_i's modulo m. Clearly

$$L(m)\le |H|\le \left(1+\frac{3}{K}\right)m+n=\frac{n+3}{n^2}m\sim\frac{m}{n}\sim\frac{m}{k}$$

as wanted. This shows that $L(m)\sim m/A(m)$, and an appeal to Theorem 3 concludes the proof of Theorem 1.

5. Concluding remarks

1) I cannot decide whether the following common generalization of Theorems 1 and 2 is true: every sequence $A=\{a_1,a_2,\dots,\}$ satisfying the condition $a_{i+1}/a_i>q>1$ has an exact asymptotic complement. I can show the existence of a set satisfying $A(x)\gg\log x$ such that every additive complement satisfies $B(x)\gg x(\log\log x)/\log x$, but this set consists of comparatively short dense blocks.
2) Let A,B be additive complements and write

$$f(n)=\#\{(a,b):a\in A,b\in B,a+b=n\}.$$

Call these complements *exact on average*, if

$$\sum_{n\le N}f(n)\sim N.$$

For sets satisfying Narkiewicz's condition (1.3) this is easily seen to be equivalent to what we called exact so far. However, there are exact on average complements that do not satisfy (1.3), indeed one can have $f(n)=1$ for all n with a familiar construction using a digital representation.

Does the set of squares have an exact on average complement?

Is there a connection corresponding to Theorem 3 between this type of complement and additive completion modulo m?

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